

A PROOF OF THE THREE GEOMETRIC INEQUALITIES CONJECTURED BY YU-DONG WU AND H.M. SRIVASTAVA

ANÍBAL CORONEL[†] AND FERNANDO HUANCAS[‡]

ABSTRACT. In this short note the authors give answers to the three open problems formulated by Wu and Srivastava [*Appl. Math. Lett.* 25 (2012), 1347–1353]. We disprove the Problem 1, by showing that there exists a triangle which does not satisfies the proposed inequality. We prove the inequalities conjectured in Problems 2 and 3. Furthermore, we introduce an optimal refinement of the inequality conjectured on Problem 3.

1. INTRODUCTION AND CONJECTURES OF YU-DONG WU AND H.M. SRIVASTAVA

The geometric inequalities are relevant in several areas of the science and engineering [1, 2, 3, 4, 5, 6, 7]. The methodologies to prove the geometric inequalities is disperse see for instance [8, 9, 10, 11]. In a broad sense, there exist some of methodologies which are based on analytical methods, other in integral and differential calculus, and other on geometric methods. The methodology of this paper, in spite of the basic arguments, can be considered belongs to the analytical methods since the results are strongly dependent on the analytical methodology introduced in [5, 12].

The focus of this short note is the open problems given in [5, 12]. Indeed, we introduce some notation and then we recall the conjectures. Let us consider a triangle $\triangle ABC$ with angles A, B and C , we denote by a, b, c, s and r , the lengths of the corresponding opposite sides, the semiperimeter and the inradius, respectively. Then, using the symbol \sum to denote a cyclic sum, i. e.

$$\sum f(b, c) = f(a, b) + f(b, c) + f(c, a),$$

we have that, the following geometric inequality

$$2\sqrt{2} s \leq \sum \sqrt{a^2 + b^2} < (2 + \sqrt{2}) s - 3\sqrt{3}(2 - \sqrt{2})r, \quad (1.1)$$

holds. In (1.1), the left inequality can be proved by the Power-Mean Inequality and the right inequality was recently proved by Wu and Srivastava in [12]. It was originally proposed by Wu [13], inspired in the following inequality [13, 14]:

$$\sum \sqrt{a^2 + b^2} < (2 + \sqrt{2})s. \quad (1.2)$$

Moreover, it is known the following generalized version of (1.2):

$$\sum \sqrt[n]{a^n + b^n} < (2 + \sqrt[n]{2})s, \quad n \in \mathbb{N} - \{1\}, \quad (1.3)$$

holds true for \mathbb{N} the set of positive integers, see [15]. Then, by analogy to the extension of Ye [15], Wu and Srivastava propposed a generalization of the right inequality in [12]. More precisely, they defined the following conjecture:

Conjecture 1. *For a given triangle $\triangle ABC$, if $n \in \mathbb{N} - \{1, 2\}$, then prove or disprove the following inequality:*

$$\sum \sqrt[n]{a^n + b^n} < (2 + \sqrt[n]{2})s - 3\sqrt{3}(2 - \sqrt[n]{2})r. \quad (1.4)$$

Date: August 14, 2014.

Key words and phrases. geometric inequalities, power mean inequality, geometric inequality conjecture.

[†] GMA, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Chillán, Chile, E-mail: acoronel@ubiobio.cl.

[‡] GMA and Doctorado en Matemática Aplicada, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Chillán, Chile, E-mail: fihuanca@gmail.com.

Additionally, in [12] Wu and Srivastava conjecture the following two inequalities:

Conjecture 2. *Let a_i ($i = 1, \dots, 6$) denote the lengths of the edges of a given tetrahedron $ABCD$. Also let ρ be the inradius of the tetrahedron. Then, determine the best constant k such the following inequality holds true:*

$$\sum_{1 \leq i, j \leq 6} \sqrt{a_i^2 + b_j^2} \leq k \sum_{i=1}^6 a_i. \quad (1.5)$$

Conjecture 3. *Let us denote by k_0 denotes the best constant k for the inequality (1.5) for a given tetrahedron $ABCD$. Then, prove or disprove the following inequality*

$$\sum_{1 \leq i, j \leq 6} \sqrt{a_i^2 + b_j^2} \leq k_0 \sum_{i=1}^6 a_i - 6\sqrt{6}(2k_0 - 5\sqrt{2})\rho. \quad (1.6)$$

We note that, the inequality (1.4) looks as a nice generalization of (1.1). However, unfortunately, we show that the inequality (1.4) is not always true, see subsection 2.1. In subsection 2.2, we determine that (1.5) holds true with $k = 2 + \sqrt{2}$. Furthermore, denoting by $k_0 = 2 + \sqrt{2}$, in subsection 2.3, we prove that the following inequality

$$\sum_{1 \leq i, j \leq 6} \sqrt{a_i^2 + b_j^2} \leq k_0 \sum_{i=1}^6 a_i - 12\sqrt{3}(2 - \sqrt{2})\rho, \quad (1.7)$$

which is a refinement of (1.6). Then, the Conjecture 3 holds true but the inequality is not optimal. Hence, summarizing the contribution of this sort note we have the following theorem:

Theorem 1.1. *Let a_i ($i = 1, \dots, 6$) denote the lengths of the edges of a given tetrahedron $ABCD$. Also let ρ be the inradius of the tetrahedron. Then, the inequalities (1.5) and (1.7) holds true with $k = k_0 = 2 + \sqrt{2}$.*

2. PROOFS OF CONJECTURES

2.1. Counterexample for Conjecture 1. We given a Counterexample which proves that (1.4) does not holds true. Indeed, let us consider $n = 3$ and the right triangle $\triangle ABC$ with $a = 3$, $b = 1$ and $c = \sqrt{10}$. Then, $s = (4 + \sqrt{10})/2$, $r = 3/(4 + \sqrt{10})$ and clearly the inequality (1.4) is reversed, since

$$\begin{aligned} \sum \sqrt[3]{a^3 + b^3} &\approx 10.116536541585731 \quad \text{and} \\ (2 + \sqrt[3]{2})s - 3\sqrt{3}(2 - \sqrt[3]{2})r &\approx 10.063472825231253. \end{aligned}$$

Furthermore, if for the given triangle we define the function $g : \mathbb{N} - \{1\} \rightarrow \mathbb{R}$ as follows

$$g(n) = (2 + \sqrt[n]{2})s - 3\sqrt{3}(2 - \sqrt[n]{2})r - \sum \sqrt[n]{a^n + b^n},$$

we note that g is strictly decreasing with $g(2) > 0 > g(3)$. Thus, for the given right triangle the inequality (1.4) is false for all $n \in \mathbb{N} - \{1, 2\}$.

2.2. Proof of Conjecture 2. We apply the inequality (1.2) over each face of the tetrahedron $ABCD$ and, naturally, we get the following optimal estimates

$$\begin{aligned} &\sum_{1 \leq i, j \leq 6} \sqrt{a_i^2 + b_j^2} \\ &= \left[\sqrt{a_1^2 + a_3^2} + \sqrt{a_3^2 + a_2^2} + \sqrt{a_2^2 + a_1^2} \right] + \left[\sqrt{a_3^2 + a_4^2} + \sqrt{a_4^2 + a_5^2} + \sqrt{a_5^2 + a_3^2} \right] \\ &\quad + \left[\sqrt{a_1^2 + a_4^2} + \sqrt{a_4^2 + a_6^2} + \sqrt{a_6^2 + a_1^2} \right] + \left[\sqrt{a_6^2 + a_5^2} + \sqrt{a_5^2 + a_2^2} + \sqrt{a_2^2 + a_4^2} \right] \\ &\leq \left(1 + \frac{\sqrt{2}}{2} \right) (a_1 + a_2 + a_3) + \left(1 + \frac{\sqrt{2}}{2} \right) (a_3 + a_4 + a_5) \end{aligned}$$

$$\begin{aligned}
& + \left(1 + \frac{\sqrt{2}}{2}\right) (a_1 + a_4 + a_6) + \left(1 + \frac{\sqrt{2}}{2}\right) (a_2 + a_5 + a_6) \\
& \leq (2 + \sqrt{2}) \sum_{i=1}^6 a_i.
\end{aligned}$$

Then, we have that the inequality (1.5) holds with optimal constant $k = 2 + \sqrt{2}$.

2.3. Proof of Conjecture 3. Let us denote by $k_0 = 2 + \sqrt{2}$ and by r_i for $i = 1, \dots, 4$ the inradius of the faces of the tetrahedron $ABCD$. Then, applying the inequality (1.4) over each face of the tetrahedron $ABCD$ we find that (1.7) holds true, since

$$\begin{aligned}
\sum_{1 \leq i, j \leq 6} \sqrt{a_i^2 + b_j^2} & \leq k_0 \sum_{i=1}^6 a_i - 3\sqrt{3}(2 - \sqrt{2}) \sum_{i=1}^4 r_i \\
& \leq k_0 \sum_{i=1}^6 a_i - 12\sqrt{3}(2 - \sqrt{2})\rho.
\end{aligned}$$

Moreover, we note that

$$k_0 \sum_{i=1}^6 a_i - 12\sqrt{3}(2 - \sqrt{2})\rho \leq k_0 \sum_{i=1}^6 a_i - 6\sqrt{6}(2k_0 - 5\sqrt{2})\rho.$$

Then, the inequality (1.6) is true, but it is not optimal.

ACKNOWLEDGEMENT

We acknowledge the support of “Univesidad del Bío-Bío” (Chile) through the research projects 124109 3/R, 104709 01 F/E and 121909 GI/C.

REFERENCES

- [1] Z. Cvetkovski. *Inequalities: Theorems, Techniques and Selected Problems*. SpringerLink : Bücher. Springer, 2012.
- [2] M.J. Cloud and B.C. Drachman. *Inequalities: With Applications to Engineering*. Springer New York, 1998.
- [3] E.H. Lieb, M. Loss, and B. Ruskai. *Inequalities: Selecta of Elliott H. Lieb*. Physics and astronomy online library. U.S. Government Printing Office, 2002.
- [4] D. S. Mitrinović, J. E. Pečarić, and V. Volenec. *Recent advances in geometric inequalities*, volume 28 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989.
- [5] Themistocles M. Rassias and Hari M. Srivastava, editors. *Analytic and geometric inequalities and applications*, volume 478 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1999.
- [6] H.M. Srivastava, V. Loksha, and Yu-Dong Wu. A new refinement of the janousgmeiner inequality for a triangle. *Computers & Mathematics with Applications*, 62(5):2349 – 2353, 2011.
- [7] Yu-Dong Wu, V. Loksha, and H.M. Srivastava. Another refinement of the pólyaszegő inequality. *Computers & Mathematics with Applications*, 60(3):761 – 770, 2010.
- [8] Razvan Alin Satnoianu. A general method for establishing geometric inequalities in a triangle. *The American Mathematical Monthly*, 108(4):pp. 360–364, 2001.
- [9] Josip Peari and Sanja Varoanec. A new proof of the arithmeticthe geometric mean inequality. *Journal of Mathematical Analysis and Applications*, 215(2):577 – 578, 1997.
- [10] Lu Yang. Recent advances in automated theorem proving on inequalities. *Journal of Computer Science and Technology*, 14(5):434–446, 1999.
- [11] Shi-Chang Shi and Yu-Dong Wu. An artificial proof of a geometric inequality in a triangle. *Journal of Inequalities and Applications*, 2013(1):329, 2013.
- [12] Yu-Dong Wu and H. M. Srivastava. An analytical proof of a certain geometric inequality conjecture of Shan-He Wu. *Appl. Math. Lett.*, 25(10):1347–1353, 2012.
- [13] S.-H. Wu. Sharpening on a geometric inequality again. *Fujian Middle School Math*, (10):9–10, 2001. in Chinese.
- [14] F.-L. Zhou. The artful proofs of several inequalities. *Fujian Middle School Math*, (10):44–55, 1996. in Chinese.
- [15] J. Ye. *Tutorial to Mathematical Olympiad*. 2003. Hunan Normal University Press, Changsha.